

Acceleration of a Pulled Spool

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A well-known lecture demonstration¹ consists of pulling a spool by the free end of a string wrapped around its inner diameter. By pulling at different angles relative to the floor, the spool can be made to roll either toward or away from you. This is explained by considering the torque about the point of contact O between the spool and the floor. A number of authors² have noted that for a cylinder rolling under the action of a horizontal pulling force, the frictional force can be in the direction of motion of its center of mass C . This often puzzles introductory students. It is therefore helpful to explore the kinematics for the more general case of pulling at an arbitrary angle with respect to the floor.

Pulling the Spool

For a spool whose mass distribution is sufficiently far away from the axis of symmetry, it turns out that there are two special pulling angles. In addition to the familiar angle θ_c in the forward direction at which the spool *cannot* roll without slipping, there is a second characteristic angle θ_m in the backward direction at which the spool *always* rolls without slipping. (Throughout this article, “forward” refers to the direction in which the string would unwind off the bottom of the spool.) In standard homework problems and class demos involving pulled cylinders, θ_m is not investigated.

The critical pulling angle θ_c at which the spool slips in place without rolling occurs when the line of action of the pulling force T is directed through O , as illustrated in Fig. 1. All other forces (the normal force N , the spool’s weight mg , and the frictional force f) also have lines of action through O ; thus, there is no net torque τ

about O and the spool does not rotate about this point. Let the inner radius of the spool (about which the string is wrapped) be R_1 and its outer radius (with which it contacts the floor) be R_2 . From the geometry it follows that $\theta_c = R_1/R_2$. For example, if $R_1 = 0.75 R_2$ then $\theta_c = 41^\circ$.

Now consider pulling at a steeper angle,³ $\theta > \theta_c$. A free-body diagram for the case in which the spool rolls without slipping (so that

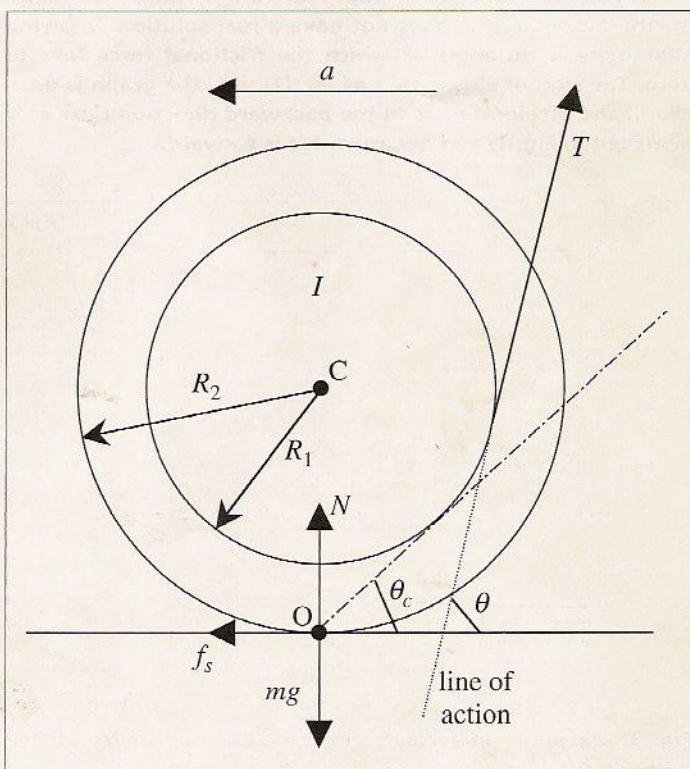


Fig. 1. Free-body diagram of a spool of mass m and moment of inertia I about its center C . A constant tension force T has a line of action making an angle θ with respect to the forward horizontal direction (which is to the right in this sketch). The directions of the static frictional force f_s and of the linear acceleration of the center of mass a follow from the equations of motion and depend on θ , as discussed in the text.

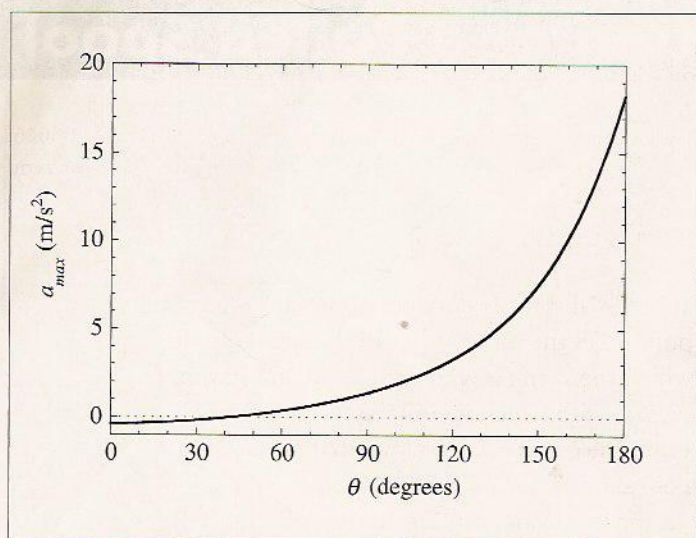


Fig. 2. Maximum linear acceleration of a spool if it is to roll without slipping when $\mu_s = 0.2$, $R_1/R_2 = 0.75$, and $\gamma = 1$. With these parameters, $\theta_c = 41^\circ$, at which angle $a_{\max} = 0$ because it is impossible for the spool to roll without slipping. Furthermore Eq. (10) does not have a real solution, implying that there is no angle at which the frictional force falls to zero. The sign of $a_{\max}$ is that of Eq. (7), i.e., the graph is positive if the acceleration is in the backward direction (just as is sketched in Fig. 1) and negative if it is forward.

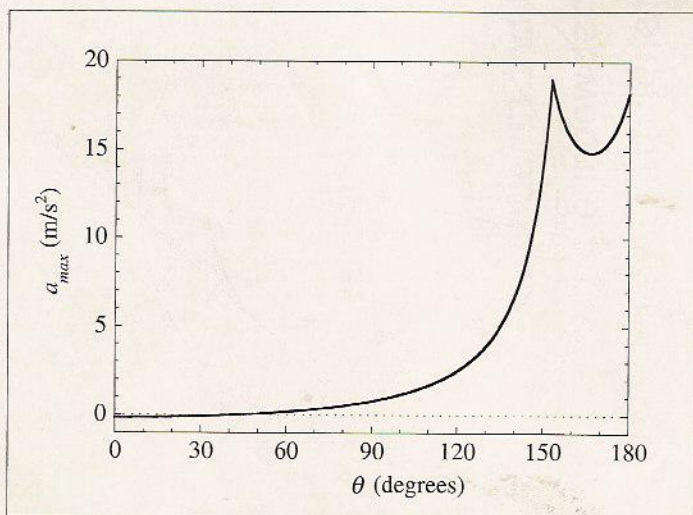


Fig. 3. Maximum linear acceleration of a spool if it is to roll without slipping when $\mu_s = 0.1$, $R_1/R_2 = 0.75$, and $\gamma = 1.5$. For these values, $\theta_c = 41^\circ$, as in Fig. 2, while $\theta_m = 153^\circ$. At this latter angle, the frictional force is zero and the spool lifts off the ground when pulled with greater than the maximum tension given by Eq. (6). Again, the acceleration is plotted with a positive sign if it is in the backward direction, and negative if forward.

the friction is static, f_s) is sketched in Fig. 1. The directions of the friction and acceleration vectors are correct as drawn provided that $\theta_c < \theta < \theta_m$, where θ_m is some maximum angle whose value will be calculated below. Applying Newton's second law horizontally gives

$$f_s - T \cos \theta = ma, \quad (1)$$

from which we see indeed that the frictional force must point in the same direction as the horizontal linear acceleration a . The rotational analog of Newton's second law counterclockwise about point C is

$$\tau = TR_1 - f_s R_2 = I\alpha = Ia/R_2, \quad (2)$$

where α is the angular acceleration of the spool and I is its moment of inertia about C. Substituting f_s from Eq. (1) into (2) gives the linear acceleration

$$a = \beta \frac{T}{m} \left(\frac{R_1}{R_2} - \cos \theta \right), \quad (3)$$

where I have defined

$$\beta \equiv [1 + I/mR_2^2]^{-1} = [1 + \gamma(R_1/R_2)^2]^{-1} \quad (4)$$

with $I = \gamma m R_1^2$. Putting $\gamma = 1$, assuming the spool's mass is effectively concentrated at its inner radius,⁴ and continuing to use the preceding value of θ_c , implies that $\beta = 0.64$. Note that a is necessarily positive because $\theta > \theta_c \Rightarrow \cos \theta < R_1/R_2$. However there is an upper limit to how hard one can pull on the string, beyond which the spool will begin to slip. This occurs when the static friction attains its maximum value,

$$f_{s,\max} = \mu_s N = \mu_s (mg - T \sin \theta), \quad (5)$$

from the vertical force balance, where μ_s is the coefficient of static friction. Therefore, the maximum allowed tension (assuming the string does not break) is

$$T_{\max} = \frac{\mu_s mg}{\cos \theta + \mu_s \sin \theta + \beta(R_1/R_2 - \cos \theta)}, \quad (6)$$

where use was made of Eqs. (1) and (3). Substituting this into Eq. (3) gives an expression for the maximum acceleration if the spool is to not slip,

$$a_{\max} = \frac{\mu_s g}{1 + \frac{\cos\theta + \mu_s \sin\theta}{\beta(R_1/R_2 - \cos\theta)}}. \quad (7)$$

This is plotted in Fig. 2 for the case of $\mu_s = 0.2$ and, as above, $\gamma = 1$ and $R_1/R_2 = 0.75$. As expected from the preceding discussion, $a_{\max} = 0$ at $\theta = \theta_c$. Thus, if the tension is smaller than the value given by Eq. (6), the spool will not move at this angle.⁵

Repeating the analysis for a shallow pulling angle, $\theta < \theta_c$, we get the same free-body diagram as in Fig. 1, except that now the acceleration points to the right rather than to the left. The friction must be the source of the clockwise torque about C, and must therefore continue to point to the left in the diagram. Hence the acceleration and its maximum value are given by the negatives of Eqs. (3) and (7), respectively. Once again a must be positive, because $\theta < \theta_c \Rightarrow \cos\theta > R_1/R_2$. Nevertheless I have plotted it as being negative on the graph in Fig. 2 to show that the maximum acceleration varies smoothly as θ crosses through θ_c .

We see from Fig. 2 that in the forward direction, the magnitude of the maximum acceleration peaks at $\theta = 0^\circ$ with the value

$$a_{\max}(0^\circ) = \mu_s g \frac{R_2/R_1 - 1}{1 + \gamma R_1/R_2} \quad (8)$$

(after properly correcting the sign). This is well behaved for any meaningful values of the parameters. In the backward direction, the acceleration monotonically rises to a peak at $\theta = 180^\circ$,

$$a_{\max}(180^\circ) = \mu_s g \frac{R_2/R_1 + 1}{1 - \gamma R_1/R_2}. \quad (9)$$

However, this expression is not always well behaved: it diverges if the denominator is zero, when $\gamma = R_2/R_1$. This can happen — an upper limit on γ is $(R_2/R_1)^2$, whereby all of the mass of the system is concentrated at the outer radius of

the spool. Note that $\gamma = R_2/R_1$ implies that the radius of gyration of the spool, $R \equiv \sqrt{I/m}$, equals the geometric mean of R_1 and R_2 , namely $\bar{R} \equiv \sqrt{R_1 R_2}$. In this case the frictional force is zero if you pull at 180° . For larger values of γ , the frictional force reverses and points in the forward direction at sufficiently large pulling angles.⁶

Specifically, let us suppose that $\gamma > R_2/R_1$ and find the angle θ_m at which the frictional force falls to zero. From Eqs. (1) and (2), this is given by

$$\theta_m = \cos^{-1}\left(-\frac{R_2}{\gamma R_1}\right), \quad (10)$$

which accordingly equals 180° when $\gamma = R_2/R_1$. At θ_m , the spool rolls frictionlessly without slipping *regardless* of how hard you pull on the string, up to the limiting value $T_{\max}$. But what happens when this value of the tension is attained? By definition, $f_s = f_{s,\max}$ at that point. But $f_s = 0$ at $\theta = \theta_m$. Hence $f_{s,\max} = \mu_s N = 0$. That is, the spool jumps off the ground. In fact, it is straightforward to show from Eqs. (7) and (10) that

$$a_{\max}(\theta_m) = \frac{-g}{\tan\theta_m}. \quad (11)$$

This does not diverge, provided $\theta_m < 180^\circ$. (Only for the special case of $\theta_m = 180^\circ$ is it impossible to jerk the spool off of the ground.) Consequently there is no discontinuity in the graph of the maximum acceleration at $\theta = \theta_m$ (except for the unrealistic case of exactly zero coefficient of static friction, as discussed on the following page).

Continuing, we can consider what happens at larger angles, $\theta > \theta_m$. The direction of the frictional force can be reversed by replacing μ_s with $-\mu_s$ in Eqs. (6), (7), and (9). The results are plotted in Fig. 3 for $\mu_s = 0.1$, $R_1/R_2 = 0.75$, and $\gamma = 1.5$ (in which case $\theta_m = 153^\circ$). There is a cusp at θ_m with a change in slope of

$$\frac{da_{\max}(\theta_m^-)}{d\theta} - \frac{da_{\max}(\theta_m^+)}{d\theta} = \frac{2(1-\beta)a_{\max}(\theta_m)}{\mu_s} \quad (12)$$

in $\text{m} \cdot \text{s}^{-2} \cdot \text{rad}^{-1}$, which is always positive since $\beta < 1$ and $a_{\text{max}}(\theta_m) > 0$. Note that Eq. (12) blows up as $\mu_s \rightarrow 0$. On a frictionless surface, it is impossible for the spool to roll without slipping except at $\theta = \theta_m$. That is, $a_{\text{max}} = 0$ everywhere except at θ_m where it takes on the positive value given by Eq. (11).

To summarize, a pulled spool has a linear acceleration given by Eq. (3). This is in the direction of the pull for shallow angles, crosses zero at angle θ_c with the horizontal, and is in the backward direction for larger angles up to 180° . The spool will roll without slipping provided the pulling force is less than that given by Eq. (6) or, equivalently, provided the magnitude of the acceleration does not exceed the absolute value of Eq. (7). If the radius of gyration of the spool is larger than the geometric mean of its inner and outer radii, then there is a second special pulling angle θ_m in the backward direction for which the spool rolls frictionlessly, provided the tension is less than $mg \csc \theta_m$. Larger pulling forces cause the spool to hop into the air at this angle. The maximum acceleration for rolling without slipping is given by Eq. (7) at all angles (to within the sign of μ_s) and is continuous (assuming a nonzero coefficient of static friction) as the angle increases through θ_m , albeit with a cusp there, the discontinuity in the first derivative being described by Eq. (12). If the spool is sequentially placed on smoother and smoother surfaces so that the frictional coefficient systematically decreases, the maximum acceleration for rolling without slipping will be found to decrease at all angles except θ_c and θ_m . These two angles are defined for an appropriate spool (i.e., one for which $I > mR_1R_2$) to be:

- (i) $\theta_c = \cos^{-1}[(\bar{R}/R_2)^2]$ with $0^\circ < \theta_c < 90^\circ$ at which the spool rotationally slips in place without rolling regardless of how large the coefficient of static friction is; and
- (ii) $\theta_m = \cos^{-1}[-(\bar{R}/R)^2]$ with $90^\circ < \theta_m < 180^\circ$ at which the spool rolls without slipping regardless of how small the coefficient of static friction is.

In the Classroom

These results can be applied to an introductory physics course in two ways. On the one hand, a three-part homework problem can be constructed. The students are given the free-body diagram in Fig. 1. In the first step, the students are asked to write down the horizontal component and rotational analog of Newton's second law using only the symbols in the free-body diagram. In the second part, the students are to solve these two equations to find the special angle θ_m at which the frictional force falls to zero. Finally, they should be told to observe from their result that this angle is unphysical unless I is larger than some minimum value and then asked what this limiting value is in terms of the mass and radii of the spool.

Secondly, a simple demonstration of the angle θ_m brings the concept home. A track is attached to the top of a pair of low-friction rolling carts (one supporting each end) traveling together on a level rail. The spool is placed near the middle of the track so that it can roll either way along it. By Newton's third law, the frictional force of the track on the spool is equal and opposite to the horizontal reaction force of the spool on the track. Hence the direction of the frictional force on the spool can be determined by observing the recoil of the carts. A wide ribbon is used to pull the spool, to help it travel in a straight line. In order to ensure that $\gamma > R_2/R_1$, a spool with a small ratio of inner to outer radii is necessary. Weights can be added near the outer rim of the spool to further increase γ —these should be added symmetrically about the axis and to both rims to maintain the balance of the spool. By measuring the masses and dimensions of the weights and component parts of the spool, θ_m can be calculated from Eq. (10). By trial and error, I found that a value of θ_m of about 120° works well: pulling the ribbon at 90° then causes the carts to recoil forward, while pulling (more sharply) at 180° produces a recoil in the other direction. (It helps to increase μ_s by adding a bit of a tacky substance to the portions of the spool that contact the track.) With some care, one can

obtain recoil-less motion of the spool to within 10° of the predicted angle. This is particularly effective as a follow-up illustration to the above homework problem.

Acknowledgment

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References

1. For example, see R. Ehrlich, *Why Toast Lands Jelly-Side Down* (Princeton University Press, NJ, 1997), p. 65.
2. A. Pinto and M. Fiolhais, "Rolling cylinder on a horizontal plane," *Phys. Educ.* **36**, 250 (May 2001) and references therein.
3. In a practical demonstration, pulling at an angle much beyond 180° is impossible unless the string passes through a groove cut in the track on which the spool rolls. For simplicity, in this article I will restrict θ to the range from 0° (forward) to 180° (backward).
4. The wound string should be included in the determinations of m and I . However, the linear mass density of the string is assumed to be small enough that the length of string pulled off the spool has negligible mass compared to γm .
5. An automated apparatus for demonstrating stable mechanical equilibrium at θ_c is characterized in P. Chagnon, "Animated displays III: Mechanical puzzles," *Phys. Teach.* **31**, 32 (Jan. 1993).
6. Forward, zero, and backward directions of the frictional force at a 180° pulling angle can be nicely demonstrated using the three rollers described in R.E. Berg, "Traction force on accelerated rolling bodies," *Phys. Teach.* **28**, 600 (Dec. 1990).